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A Holomorphic Version of Landau's Theorem

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INTRODUCTION

In this paper I consider variations on a theme of Edmund Landau: namely, if a function and its nth derivative ($n \geqslant 2$) are bounded, say on the real line, then so are all the intermediate derivatives and bounds can be obtained relating the size of the rth derivative, $0 < r < n$, to those of the 0th and nth derivatives; further, the function which has the largest rth derivative is (essentially) unique and possesses a number of interesting properties. See [2] for references to Landau's work.

The problems considered here are modifications of the following. Let Δ_0 be the open unit disc in the complex plane, let n be an integer $\geqslant 2$, let $r \in \{1,..., n-1\}$, let K be a compact subset of $\Delta = \{z: |z| \leqslant 1\}$ with n or more points and let σ be a positive number. The problem is to find among those functions f which are holomorphic on Δ_0 and which satisfy

$$\max\{|f(z)|: z \in K\} \leqslant 1, \qquad \sup_{z \in \Delta_0} |f^{(n)}(z)| \leqslant \sigma, \tag{1}$$

one for which

$$\max_{z \in \Delta} |f^{(r)}(z)| \tag{2}$$

is as large as possible. Since $f^{(r)}$ is continuous on Δ, the problem above is equivalent to this one. Let $\xi \in \partial\Delta = T = \{|z| = 1\}$ be fixed. Find those functions F satisfying (1), for which $|F^{(r)}(\xi)|$ equals

$$\gamma(\sigma) =: \max\{|f^{(r)}(\xi)|: f \text{ satisfies } (1)\}. \tag{2'}$$

Of course, γ depends on K, ξ, n and r, as well as σ, but I suppress this

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dependence in my notation. Before proceeding to analyze the solutions of this extremal problem, I will generalize it slightly. Let $1 < p \leqslant \infty$, and let H^p be the usual Hardy space of functions on Δ_0; see [1], for example. With n, r and σ as above, and with ξ any point of Δ which is not in the interior of the convex hull of K, the new extremal problem is this: find among those functions, holomorphic in Δ_0 and satisfying

$$\max_{z \in K} |f(z)| \leqslant 1, \qquad f^{(n)} \in H^p \quad \text{and} \quad \|f^{(n)}\|_p \leqslant \sigma, \tag{1'}$$

one for which $|f^{(r)}(\xi)|$ is as large as possible. The subscript p in (1′) refers to the H^p norm of $f^{(n)}$. Let $\gamma(\sigma, p)$ denote this maximum:

$$\gamma(\sigma, p) =: \max\{|f^{(r)}(\xi)| : f \text{ satisfies } (1')\} \tag{3}$$

so that $\gamma(\sigma, \infty) = \gamma(\sigma)$. Any function F satisfying (1′) for which $F^{(r)}(\xi) = \gamma(\sigma, p)$ will be termed *extremal.* A simple normal families argument shows that there is at least one extremal function. I shall show in Section 1 that there is precisely one extremal function. In Section 2 I analyze the growth of $\gamma(\sigma, p)$, as a function of σ, when $\sigma \to \infty$, in relation to the set K. Finally, in Section 3, I describe a few properties of the extremal function.

1. Uniqueness

Define X to be those holomorphic functions f on Δ_0 for which $f^{(n)} \in H^p$, and define a norm on X by

$$\|f\| = \max \left\{ \|f\|_K, \frac{1}{\sigma} \|f^{(n)}\|_p \right\}, \tag{4}$$

where

$$\|f\|_K = \max\{|f(z)| : z \in K\}.$$

With this norm, X is a Banach space and the functions satisfying (1′) are precisely the unit ball of X. Hence, $\gamma(\sigma, p)$ is the norm of the linear functional on X given by $l_0(f) = f^{(r)}(\xi)$. The extremal problem is then to determine the norm of this functional l_0 and to find those elements of X at which l_0 attains its norm. X is a closed subspace of the Banach space Y consisting of the direct sum of $C(K)$ and L^p with norm

$$\|(u, g)\| = \max \left\{ \|u\|_K, \frac{1}{\sigma} \|g\|_p \right\}$$

when we make the usual identification of H^p with the closed subspace of $L^p = L^p(T, d\theta)$ consisting of those functions whose negative Fourier coefficients vanish; see [1]. The dual space of Y is the direct sum of $\mathscr{M}(K)$, the finite regular Borel measures on K, and $L^{p'}$, where p' is the conjugate exponent of p, with the norm

$$\|(\mu, h)\| = \|\mu\| + \sigma \|h\|_{p'}. \tag{5}$$

Basic duality for Banach spaces then implies

$$\begin{aligned} \gamma(\sigma, p) &= \inf\{\|l\| : l \in Y^*, l = l_0 \text{ on } X\} \\ &= \inf\{\|l_0 + m\| : m \in Y^*, m \perp X\}. \end{aligned} \tag{6}$$

Now if $f \in X$, then $f(z) = \sum_{j=0}^{\infty} a_j z^j$ for $z \in \Delta$ and so

$$\begin{aligned} l_0(f) &= f^{(r)}(\xi) \\ &= \sum_{s=r}^{\infty} a_s \frac{s!}{(s-r)!} \xi^{s-r} \\ &= \sum_{s=r}^{n-1} a_s \frac{s!}{(s-r)!} \xi^{s-r} + \int_T f^{(n)}(e^{i\theta})\, G(e^{i\theta})\, d\theta, \end{aligned}$$

where

$$G(e^{i\theta}) = \sum_{j=0}^{\infty} \frac{j!}{(j+n-r)!} \xi^{j+n-r} e^{-ij\theta}. \tag{7}$$

Note that G lies in L^p for all finite p, even if $r = n-1$ and $|\xi| = 1$. If μ is a measure on K, define

$$(L\mu)(e^{i\theta}) =: -\sum_{0}^{\infty} \frac{j!}{(j+n)!} \left\{ \int_K z^{j+n}\, d\mu(z) \right\} e^{-ij\theta}. \tag{8}$$

There are two immediate consequences of (8). The first is that each pair $(\mu, v) \in Y^*$ which annihilates X has the form $(\mu, L\mu + h)$ where $h \in H_0^{p'}$, and μ satisfies

$$\int_K z^s\, d\mu(z) = 0 \qquad \text{for} \quad s = 0, \ldots, n-1. \tag{9}$$

($H_0^{p'}$ consists of those $H^{p'}$ functions with mean-value zero.) The second is that if

$$\int_K z^s\, d\mu(z) = \begin{cases} 0, & 0 \leqslant s < r, \\ \dfrac{s!}{(s-r)!}\xi^{s-r}, & r \leqslant s \leqslant n-1, \end{cases} \tag{10}$$

then

$$\begin{aligned} \int_K f\, d\mu + \int_T f^{(n)}(L\mu + G)\, d\theta &= \sum_{s=r}^{\infty} a_s \frac{s!}{(s-r)!}\xi^{s-r}, \qquad f \in X, \\ &= f^{(r)}(\xi). \end{aligned} \tag{11}$$

Set

$$\Lambda = \{\mu \in \mathscr{M}(K) \colon (10) \text{ holds}\}. \tag{12}$$

Then formulas (5)–(7), (9)–(11) imply the important relation

$$\gamma(\sigma, p) = \inf\{\|\mu\|_{\mathscr{M}} + \sigma\, \|G + L\mu\|_{L^{p'}/H_0^{p'}} \colon \mu \in \Lambda\}. \tag{13}$$

Here $L^{p'}/H_0^{p'}$ is the usual quotient space of $L^{p'}$ by $H_0^{p'}$. Formula (13) will be the basis for much of what follows.

The linear transformation L carries $\mathscr{M}(K)$ into $C(T)$ and is continuous from the weak-* topology to the norm topology since $n \geqslant 2$. It follows from this and from the fact that the unit ball of $H_0^{p'}$ is weakly compact for $1 < p' < \infty$ and weak-* compact in $\mathscr{M}(T)$ if $p' = 1$, that for each $\sigma > 0$ there is at least one measure $\mu_\sigma \in \Lambda$ and at least one $h_\sigma \in H_0^{p'}$ for which equality holds in (13):

$$\gamma(\sigma, p) = \|\mu_\sigma\|_{\mathscr{M}} + \sigma\, \|L\mu_\sigma + G + h_\sigma\|_{p'}.$$

Now let F_σ be an extremal function. Then

$$\begin{aligned} \gamma(\sigma, p) &= F_\sigma^{(r)}(\xi) \\ &= \int_K F_\sigma\, d\mu_\sigma + \int_T F_\sigma^{(n)}(L\mu_\sigma + G + h_\sigma)\, d\theta \\ &\leqslant \|\mu_\sigma\| + \sigma\, \|L\mu_\sigma + G + h_\sigma\|_{p'} \\ &= \gamma(\sigma, p). \end{aligned}$$

Consequently, equality holds throughout and we learn that

(a) $|F_\sigma| = 1$ on $\operatorname{supp}(\mu_0)$,

(b) $F_\sigma\, d\mu_\sigma$ is a non-negative measure. (14)

Further,

$$F_\sigma^{(n)}(L\mu_\sigma + G + h_\sigma) \geqslant 0 \qquad \text{a.e. } d\theta, \tag{15}$$

and

$$\begin{aligned} &\text{(a)} \quad |F_\sigma^{(n)}| = 1 \qquad \text{a.e. } d\theta \text{ where } L\mu_\sigma + G + h_\sigma \neq 0 \text{ if } p = \infty, \\ &\text{(b)} \quad |F_\sigma^{(n)}|^p = c\,|L\mu_\sigma + G + h_\sigma|^{p'} \qquad \text{a.e. } d\theta \text{ if } 1 < p < \infty, \end{aligned} \tag{16}$$

where $c = \sigma^p / \|L\mu_\sigma + G + h_\sigma\|^{p'}$.

In (16)(a) if $L\mu_\sigma + G + h_\sigma = 0$ a.e. $d\theta$, then $L\mu_\sigma + G = 0$ a.e. $d\theta$ since $L\mu_\sigma + G$ is the conjugate of an element of H^2. However, this would imply that

$$\int_K z^{j+n}\, d\mu_\sigma(z) = \frac{(j+n)!}{(j+n-r)!}\, \xi^{j+n-r} \qquad \text{for} \quad j = 0, 1, \dots,$$

so that

$$\int_K f\, d\mu_\sigma = f^{(r)}(\xi) \qquad \text{for all } f \in X, \tag{17}$$

and, in particular, for all functions holomorphic in a neighborhood of Δ. Since ξ does *not* lie in the interior of the convex hull of K, there is a sequence $\{f_n\}$ of polynomials for which $f_n \to 0$ uniformly on K but $|f_n^{(r)}(\xi)| \to \infty$, a contradiction to (17). Thus, $L\mu_\sigma + G + h_\sigma \neq 0$ on a set $\mathscr{E}$ of positive Lebesgue measure in T. If H is another extremal function, then so is $\frac{1}{2}(F_\sigma + H)$ and so all the conclusions in (14), (15), (16) apply to H and to $\frac{1}{2}(F_\sigma + H)$. Thus $F_\sigma^{(n)} = H^{(n)}$ a.e. on T if $1 < p < \infty$ by (15) or $F_\sigma^{(n)} = H^{(n)}$ a.e. on $\mathscr{E}$ if $p = \infty$ by (16), and hence $F_\sigma^{(n)} = H^{(n)}$ a.e. on T. In either case, $F_\sigma - H$ is a polynomial of degree $n-1$ or less. However, (14)(a) implies $F_\sigma = H$ on the support of μ_σ. I show below that μ_σ has n or more points in its support; this implies immediately that $F_\sigma \equiv H$. To see that μ_σ has n or more points in its support, note that $\int p\, d\mu_\sigma = p^{(r)}(\xi)$ for all polynomials p of degree $n-1$ or less. If $\operatorname{supp}(\mu_\sigma) = \{\zeta_1, \dots, \zeta_s\}$, where $s \leqslant n-1$, set $P(z) = \prod_1^s (z - \zeta_j)$. Then $P = 0$ on support μ_σ but $P^{(r)}(\xi) \neq 0$ by the Gauss–Lucas theorem (recall ξ does not lie in the interior of the convex hull of K). This completes the proof of uniqueness.

I summarize the results of this section.

THEOREM 1. *There is precisely one function F satisfying* (1′) *with*

$$F^{(r)}(\xi) = \max\{|f^{(r)}(\xi)| : f \text{ satisfies } (1')\}.$$

COROLLARY 2. *Suppose K is symmetric with respect to the real axis and ξ is real. Then F_σ is real on the real axis.*

Proof: $G(z) = F_\sigma(\bar{z})$ is another extremal function and hence coincides with F_σ.

2. The Dependence of $\gamma(\sigma)$ on σ and K

I begin with a look at the measure μ_σ.

DEFINITION. Let $p \in (1, \infty]$ be fixed. A measure $\mu \in \Lambda$ for which equality holds in (13) will be termed extremal. That is, μ is extremal if

$$\int_K z^s \, d\mu(z) = \begin{cases} 0, & 0 \leqslant s < r, \\ \dfrac{s!}{(s-r)!} \xi^{s-r}, & r \leqslant s < n, \end{cases} \tag{18}$$

and

$$\gamma(\sigma, p) = \|\mu\| + \sigma \|L\mu + G\|_{L^{p'}/H_0^{p'}}. \tag{19}$$

PROPOSITION 3. *If* $1 < p < \infty$, *then there is precisely one extremal measure.*

Proof: Let μ and ν be extremal measures. Then $\beta = \frac{1}{2}(\mu + \nu)$ lies in Λ so that

$$\begin{aligned} \gamma(\sigma, p) &\leqslant \|\beta\| + \sigma \|L\beta + G\| \\ &\leqslant \tfrac{1}{2}\|\mu\| + \tfrac{1}{2}\|\nu\| + \tfrac{1}{2}\sigma \|L\mu + G\| + \tfrac{1}{2}\sigma \|L\nu + G\| \\ &= \gamma(\sigma, p). \end{aligned}$$

Hence, because $L^{p'}/H_0^{p'}$ is uniformly convex,

$$L\mu + G = L\nu + G, \qquad \text{mod } H_0^{p'}. \tag{20}$$

Now $L\mu + G$ and $L\nu + G$ both are the complex conjugates of H^2 functions so (20) implies that $L\mu = L\nu$. Thus,

$$\int_K z^j \, d\mu(z) = \int_K z^j \, d\nu(z), \qquad j = 0, 1, 2, \ldots. \tag{21}$$

Hence, $\mu - \nu$ is orthogonal to all functions analytic on Δ. But $F_\sigma \, d\mu$ and $F_\sigma \, d\nu$ are both non-negative measures so that the real measure $F_\sigma(d\mu - d\nu)$ is orthogonal to z^j for $j = 0, 1, 2, \ldots$. Thus, this measure must vanish and so $\mu = \nu$. (We note parenthetically here that any extremal measure μ is supported on the outer boundary of K since $|F_\sigma| = 1$ on $\operatorname{supp}(\mu)$ and $1 \geqslant |F|$ on K.)

Remark. In the case when K is symmetric with respect to the real axis it is not difficult to show that for $1 < p < \infty$ the (unique) extremal measure μ_σ is also symmetric with respect to the real axis in the sense that the μ_σ-measure of a set E in K is the complex conjugate of the μ_σ-measure of the set $\{\bar{z}: z \in E\}$.

We now investigate how $\gamma(\sigma)$ behaves as a function of σ. Recall formula (12):

$$\gamma(\sigma, p) = \inf\{\|\mu\| + \sigma \, \|S\mu\|_{L^{p'}/H_0^{p'}} : \mu \in \Lambda\},$$

where

$$S\mu =: L\mu + G = \sum_{j=0}^{\infty} \frac{j!}{(j+n)} \left\{ \int_K z^{j+n} \, d\mu(z) \right\} e^{-ij\theta} + \sum_{j=0}^{\infty} \frac{j!}{(j+n-r)!} \xi^{j+n-r} e^{-ij\theta}$$

and

$$\Lambda = \left\{ \mu \in \mathscr{M}(K) : \int z^s \, d\mu(z) \right\} = \begin{cases} 0, & 0 \leqslant s < r, \\ \dfrac{s!}{(s-r)!} \xi^s, & r \leqslant s \leqslant n-1. \end{cases}$$

Note that $S(t\mu_1 + (1-t)\mu_2) = tS\mu_1 + (1-t) S\mu_2$ for any $t \in \mathbb{R}$ and that Λ is a convex set. Note further that S is continuous from Λ with the weak-* topology into $L^{p'}$ for $1 \leqslant p' \leqslant \infty$. I now write $\gamma(\sigma)$ for $\gamma(\sigma, p)$, p being fixed. As well, I shall drop the subscript $L^{p'}/H_0^{p'}$ on the norm of $S\mu$. Define

$$A(\sigma) = \inf\{\|\mu\| : \mu \in \Lambda \text{ and } \gamma(\sigma) = \|\mu\| + \sigma \, \|S\mu\|\}, \tag{22}$$

$$B(\sigma) = \frac{1}{\sigma} [\gamma(\sigma) - A(\sigma)]. \tag{23}$$

When $1 < p < \infty$, $A(\sigma) = \|\mu_\sigma\|$ since the extremal measure is unique; when $p = \infty$, $A(\sigma)$ is the smallest variation of any extremal measure.

THEOREM 4. (i) *$\gamma(\sigma)$ and $A(\sigma)$ are increasing functions of σ and $B(\sigma)$ is a decreasing function of σ.*

(ii) *$\gamma(\sigma)$ is continuous and is in* Lip 1.

(iii) *If $1 < p < \infty$, then A and B are continuous and γ is differentiable with $\gamma'(\sigma) = B(\sigma)$.*

(iv) *If* $p = \infty$, *then* A *and* B *are left continuous and* γ *is left differentiable with left derivative equal to* $B(\sigma)$.

(v) γ *is concave down.*

(vi) $\lim_{\sigma \to \infty} \gamma(\sigma)/\sigma$ *exists and equals* $\lim_{\sigma \to \infty} B(\sigma)$.

Proof. For each $t > 0$ there is a $\lambda_t \in \Lambda$ with $A(t) = \|\lambda_t\|$ and $\gamma(t) = \|\lambda_t\| + t\,\|S\lambda_t\|$. If $\delta > 0$, then

$$\begin{aligned}\gamma(t) &\leqslant \|\lambda_{t+\delta}\| + t\,\|S\lambda_{t+\delta}\| \\ &< \|\lambda_{t+\delta}\| + (t+\delta)\,\|S\lambda_{t+\delta}\| \\ &= \gamma(t+\delta).\end{aligned}$$

Hence, γ is increasing. For any real number δ, we have

$$\begin{aligned}\gamma(t) &\leqslant \|\lambda_{t+\delta}\| + t\,\|S\lambda_{t+\delta}\| \\ &= \gamma(t+\delta) - \delta\,\|S\lambda_{t+\delta}\|\end{aligned}$$

and

$$\begin{aligned}\gamma(t+\delta) &\leqslant \|\lambda_t\| + (t+\delta)\,\|S\lambda_t\| \\ &= \gamma(t) + \delta\,\|S\lambda_t\|,\end{aligned}$$

so that

$$B(t+\delta) \leqslant (\gamma(t+\delta) - \gamma(t))/\delta \leqslant B(t), \qquad \delta > 0, \tag{24}$$

and the reverse inequalities if $\delta < 0$. This shows B is decreasing. Once B is shown to be continuous (or left-continuous) then (24) will show γ is differentiable (or left-differentiable) with derivative equal to $B(\sigma)$.

The inequality

$$\gamma(t+\delta) \leqslant \gamma(t) + \delta\,\|S\lambda_t\|$$

derived above implies that

$$|\gamma(\sigma) - \gamma(\tau)| \leqslant |\sigma - \tau|\, M, \qquad \sigma, \tau > 0,$$

where $M = \max_\sigma B(\sigma)$. We note that $B(\sigma)$ is bounded for $\sigma \leqslant 1$ and hence all σ since

$$\begin{aligned}B(\sigma) = \|L\mu_\sigma + G\| &\leqslant \|G\| + \|L\|\,\|\mu_\sigma\| \\ &\leqslant \|G\| + \|L\|\,\gamma(\sigma) \\ &\leqslant \|G\| + \|L\|\,\gamma(1) = M.\end{aligned}$$

Next let t be fixed and let $\delta_j \to 0$ with

$$\lim_{j\to\infty} A(t+\delta_j) = \liminf_{s\to t} A(s)$$

and set $\lambda_j = \lambda_{t+\delta_j}$. Since $\{\|\lambda_j\|\}$ is bounded, say by $\gamma(t+1)$, we may assume that $\lambda_j \to^* \lambda$, where $\lambda \in \Lambda$. Hence, $S\lambda_j \to S\lambda$ in norm. Thus,

$$\begin{aligned}\gamma(t) &\leqslant \|\lambda\| + t\,\|S\lambda\| \\ &\leqslant \lim\inf\{\|\lambda_j\| + (t+\delta_j)\,\|S\lambda_j\| \\ &= \lim A(t+\delta_j) + (t+\delta_j)\,B(t+\delta_j) \\ &= \lim \gamma(t+\delta_j) \\ &= \gamma(t).\end{aligned}$$

Thus, $A(t) \leqslant \|\lambda\| = \lim \|\lambda_j\| = \lim A(t+\delta_j)$ so that

$$A(t) \leqslant \liminf_{s\to t} A(s).$$

Note, as well, that for $\delta > 0$ we have

$$\begin{aligned}A(t+\delta) - A(t) &= \gamma(t+\delta) - \gamma(t) - \delta B(t+\delta) + t[B(t) - B(t+\delta)] \\ &\geqslant 0\end{aligned}$$

by (24) above. Hence, A is increasing and so continuous from the left. Consequently, B is contiuous from the left as well.

In the case $1 < p < \infty$, we know that the extremal measure is unique. If $\delta_j \to 0$ with $\lim_{j\to\infty} A(t+\delta_j) = \lim\sup_{s\to t} A(s)$, then subsequence of the extremal measures $\{\lambda_{t+\delta_j}\}$ must converge weak-* to λ_t as above, by the uniqueness of λ_t. Hence,

$$A(t) = \limsup_{s\to t} A(s) \geqslant \limsup_{s\to t} A(s) \geqslant A(t).$$

For any positive numbers t and δ, we have

$$\gamma(t+\delta) \leqslant \gamma(t) + \delta B(t)$$

so that

$$\limsup_{\delta\to\infty} \frac{\gamma(t+\delta)}{t+\delta} \leqslant B(t)$$

and hence

$$\limsup_{\sigma\to\infty} \frac{\gamma(\sigma)}{\sigma} \leqslant \liminf_{\sigma\to\infty} B(\sigma).$$

However, $B(\sigma) \leqslant \gamma(\sigma)/\sigma$ for all σ so that

$$\liminf_{\sigma\to\infty} \gamma(\sigma)/\sigma \geqslant \liminf_{\sigma\to\infty} B(\sigma).$$

Thus

$$\operatorname*{limit}_{\sigma\to\infty} \frac{\gamma(\sigma)}{\sigma} = \liminf_{\sigma\to\infty} B(\sigma) = \operatorname*{limit}_{\sigma\to\infty} B(\sigma)$$

since B is decreasing.

Finally, note that for $\delta > 0$,

$$\begin{aligned}\gamma(t+\delta) - 2\gamma(t) + \gamma(t-\delta) &\leqslant \gamma(t) + \delta B(t) - 2\gamma(t) + \gamma(t) - \delta B(t)\\ &= 0,\end{aligned}$$

by (24) so that γ is concave since γ is continuous.

THEOREM 5. *Suppose K is starlike with respect to ξ. Then*

$$\gamma(\sigma)/\sigma^{r/n} \leqslant \gamma(\tau)/\tau^{r/n}, \qquad \text{for} \quad \sigma \geqslant \tau. \tag{25}$$

Proof. Let $\delta = \tau/\sigma$ and define

$$g(z) = F_\sigma((1-\delta)^{1/n} z + (1-(1-\delta)^{1/n})\,\xi), \qquad z \in \Delta.$$

Then $|g(z)| \leqslant 1$ for $z \in K$ since K is starlike with respect to ξ and

$$\| g^{(n)} \|_p \leqslant (1-\delta)\, \|F_\sigma^{(n)}\|_p = (1-\delta)\,\sigma = \sigma - \tau.$$

Hence,

$$\begin{aligned}\gamma(\sigma-\tau) &= \gamma((1-\delta)\,\sigma)\\ &\geqslant |g^{(r)}(\xi)|\\ &= (1-\delta)^{r/n}\,\gamma(\sigma).\end{aligned}$$

Thus,

$$\gamma(\sigma-\tau)/(\sigma-\tau)^{r/n} \geqslant \gamma(\sigma)/\sigma^{r/n}, \qquad \sigma > \tau,$$

which is equivalent to (25).

COROLLARY 6. *If $\xi = 1$ and K is starlike with respect to* 1, *then*

$$\sigma^{r/n} \leqslant \gamma(\sigma) \leqslant \gamma(1)\,\sigma^{r/n}, \qquad r \geqslant 1 \quad \text{and} \quad 1 < p \leqslant \infty. \tag{26}$$

Proof. Set $f(z) = \exp[\sigma^{1/n}(z-1)]$; then $\|f\| \leqslant 1$ on all of Δ, $\|f^{(n)}\|_p \leqslant \sigma$ for all p. Hence,

$$\gamma(\sigma) \geqslant |f^{(r)}(1)| = \sigma^{r/n}.$$

The next theorem treats the case when K has only a finite number of points and shows a strong contrast to the case just covered.

THEOREM 7. *Suppose K has a finite number of points. Then*

(i) $A =: \lim_{\sigma\to\infty} A(\sigma)$ *is finite.*
(ii) $B =: \lim_{\sigma\to\infty} B(\sigma)$ *is positive.*
(iii) $0 \leqslant [A + \sigma B] - \gamma(\sigma) \to 0$ *as* $\sigma \to \infty$.
(iv) $B = \inf\{\|L\mu + G\|_{L^{p'}/H^{p'}} : \mu \in \Lambda\}$.

Proof. Because K is finite, $\mathscr{M}(K)$ is finite-dimensional. The operator L is clearly one-to-one on $\mathscr{M}(K)$, again because K is finite, so that L is a homeomorphism. Thus, the range of L on Λ is closed and since G is not in the set $L(\Lambda)$, the distance from G to $L(\Lambda)$ must be positive; this proves (ii).

To see that (i) holds, suppose that u is any element of $C(K)$ with sup norm 1. If K has N points, there is a polynomial f of degree $N-1$ with $f = u$ on K. Let $Pf(z) = \sum_0^{n-1} (f^{(k)}(0)/k!)\, z^k$; then

$$\begin{aligned}
\int_K u\, d\mu_\sigma &= \int_K f\, d\mu_0 \\
&= \int_K Pf\, d\mu_\sigma + \int_K (f - Pf)\, d\mu_\sigma \\
&= (Pf)^{(r)}(\xi) + \int_T f^{(n)} L\mu_\sigma \\
&= (Pf)^{(r)}(\xi) + \int_T f^{(n)}(L\mu_\sigma + G + h_\sigma) - \int_T f^{(n)} G.
\end{aligned}$$

Each of the three terms above remains bounded as $\sigma \to \infty$ and hence the uniform boundedness principle implies that $\|\mu_\sigma\| \leqslant M$ for all σ; this proves (i).

Since (i) holds some subsequence of $\{\mu_\sigma\}$ converges weak-*, and hence in norm since K is finite, to a measure μ_∞ which lies in Λ and which satisfies $A = \|\mu_\infty\|$ and $B = \|L\mu_\infty + G\|$. Consequently,

$$-\gamma(\sigma) \geqslant -\|\mu_\infty\| - \sigma\|L\mu_\infty + G\|$$
$$\gamma(\sigma) = \|\mu_\infty\| + \sigma\|L\mu_\infty + G\|$$

so that $A - A(\sigma) \geqslant \sigma(B(\sigma) - B)$ which implies

$$[A + B\sigma] - \gamma(\sigma) = A - A(\sigma) + \sigma(B - B(\sigma)) \geqslant 0$$

and

$$[A + B\sigma] - \gamma(\sigma) \leqslant A - A(\sigma) + \sigma(B(\sigma) - B)$$
$$\leqslant 2(A - A(\sigma)) \to 0.$$

This establishes (iii). To see that (iv) holds, suppose that $\lambda \in \Lambda$ and $B - \delta = \|L\lambda + G\|$ for some $\delta > 0$. Then

$$\gamma(\sigma)/\sigma \leqslant \|\lambda\|/\sigma + \|L\lambda + G\|$$
$$= \|\lambda\|/\sigma + B - \delta$$

for all $\sigma > 0$. Let $\sigma \to \infty$ and use (vi) of Theorem 4 to reach a contradiction.

THEOREM 8. *Let K have precisely n points. Then*

$$F_\sigma = Q + \sigma H, \qquad \textit{for all } \sigma > 0, \tag{27}$$

where Q is a polynomial of degree $n-1$ or less and H is an element of X, $H \equiv 0$ on K.

Proof. If K has n points, then Λ is a singleton, say $\Lambda = \{\mu\}$. Hence,

$$\frac{1}{\sigma} F_\sigma^{(n)} = \begin{cases} \operatorname{sgn}(L\mu + G + h)\,|L\mu + G + h|^{p'-1}, & 1 < p < \infty, \\ \operatorname{sgn}(L\mu + G + h), & p = \infty, \end{cases}$$

for all σ. Thus, $F_\sigma = Q_\sigma + \sigma H$, where Q_σ is a polynomial of degree $n-1$ or less and H vanishes on K. But $F_\tau = F_\sigma$ on $\operatorname{supp}(\mu)$ by (14) so that $Q_\sigma = Q_\tau$ at n points and hence $Q_\sigma \equiv Q_\tau =: Q$. Thus,

$$F_\sigma = Q + \sigma H.$$

COROLLARY 9. *If K has precisely n points, then $\gamma(\sigma) = A + B\sigma$ where A, B are constants.*

Remarks. (1) The polynomial Q in Theorem 8 is the solution to extremal problem (2) for $\sigma = 0$; that is, Q has maximal rth derivative at ξ among all polynomials of degree $n-1$ which are bounded by 1 in modulus on K. Furthermore, H is the solution of the extremal problem described by maximizing $h^{(r)}(\xi)$ under the restrictions that $\|h^{(n)}\|_p = 1$ and $h = 0$ on the set K.

(2) The case when K is finite constrasts strongly with the case when say, $K = \Delta$. In the former case, the growth of $\gamma(\sigma)$ is basically the same for all r and n (the constants A and B depend on r and n, however), whereas in the latter case, the growth of $\gamma(\sigma)$ is basically $\sigma^{r/n}$ and thus depends quite directly on r and n.

EXAMPLE 10. Let us take n to be 3, $K = \{1, \lambda, \lambda^2\}$ where $\lambda = \exp[2\pi i/3]$ and $\xi = 1$. Then by Theorem 8,

$$F_\sigma = Q + \sigma H,$$

where Q is a polynomial of degree 2, $H = 0$ on K, and $\|H^{(3)}\|_p = 1$. Q must be the unique polynomial of degree 2 which is bounded by 1 in modulus on K and which has maximal rth derivative at 1 among all such polynomials. First consider the case $r = 1$. Here the unique element μ_1 of Λ has weights 1, $\lambda(1-\lambda)^{-1}$, and $(\lambda - 1)^{-1}$ at $1, \lambda$, and λ^2, respectively. Let $Q_1(z)$ be defined by

$$3Q_1(z) = (1 - \sqrt{3}) + z + (1 + \sqrt{3})\, z^2.$$

Then $|Q_1| = 1$ on K and, indeed,

$$Q_1(\lambda^k)\,\mu_1(\{\lambda^k\}) = |\mu_1(\{\lambda^k\})|, \qquad k = 0, 1, 2,$$

so that Q_1 is the extremal polynomial for $r = 1$.

Next consider the case $r = 2$. The unique element μ_2 of Λ has weights 2/3, $(-2/3)(1 + \lambda^2)$, and $(2/3)(\lambda^2)$ at 1, λ, and λ^2 respectively, and

$$Q_2(z) = z^2$$

is the extremal polynomial for $r = 2$. It follows that for all p, $1 < p \leqslant \infty$, and all $\sigma \geqslant 0$, the extremal function for $r = 1$ is not the same as the extremal function for $r = 2$.

When $p = 2$, the best H_0^2 approximation to $L\mu + G$ is zero. Hence,

$$H^{(3)}(e^{i\theta}) = \overline{c(L\mu(e^{i\theta}) + G(e^{i\theta}))}$$

where c is a constant selected so that $\|H^{(3)}\|_2 = 1$. A computation of the Fourier coefficients of μ_1 and then of μ_2 yields, for $r = 1$ and $r = 2$,

$$H_1^{(3)}(e^{i\theta}) = c_1[-g_1(e^{i\theta}) - 2g_2(e^{i\theta}) + G_1(e^{i\theta})] \tag{28}$$

and

$$H_2^{(3)}(e^{i\theta}) = c_2[-2g_2(e^{i\theta}) + G_2(e^{i\theta})], \tag{29}$$

where

$$g_1(e^{i\theta}) = \sum_0^\infty [(3k+1)!/(3k+4)!]\, e^{(3k+1)i\theta}, \tag{30}$$

$$g_2(e^{i\theta}) = \sum_0^\infty [(3k+2)!/(3k+5)!]\, e^{(3k+2)i\theta}, \tag{31}$$

and

$$G_1(e^{i\theta}) = \sum_0^\infty [k!/(k+2)!]\, e^{ik\theta}, \tag{32}$$

$$G_2(e^{i\theta}) = \sum_0^\infty [k!/(k+1)!]\, e^{ik\theta}. \tag{33}$$

Note that

$$(z^3 g_1))''' = z(1-z^3)^{-1},$$
$$(z^3 g_2(z))''' = z^2(1-z^3)^{-1},$$

which shows that g_1, g_2 have analytic extensions to the complex plane with the three rays $\{t\lambda^k : t \geqslant 1\}$, $k = 0, 1, 2$, deleted. As well, G_1 and G_2 have analytic extensions to the complex plane with the ray $\{t : t \geqslant 1\}$ deleted. Formulas (28)–(33) completely describe H_1 and H_2 along with the fact that H_1 and H_2 both vanish on K.

3. Properties of F_σ

We begin by analyzing the operator L given in (8).

Proposition 11. *Let L be defined on $\mathscr{M}(K)$ by*

$$(L\mu)(e^{i\theta}) = -\sum_{j=0}^\infty (j!/(j+n)!) \left\{\int_K z^{j+n}\, d\mu(z)\right\} e^{-ij\theta}. \tag{34}$$

Then

$$(L\mu)(e^{i\theta}) = -e^{in\theta} \int_K M_n(ze^{-i\theta})\, d\mu(z), \tag{35}$$

where

$$M_n(w) =: \sum_{j=0}^{\infty} (j!/(j+n)!\, w^{j+n}, \qquad |w| \leqslant 1, \tag{36}$$
$$= A_n(1-w)^{n-1}(\log(1-w) + B_n),$$

and

$$A_n = (-1)^n/(n-1)!, \tag{37}$$

$$B_n = \begin{cases} 0, & n = 1, \\ -\sum_{1}^{n-1} 1/j, & n \geqslant 2. \end{cases} \tag{38}$$

Proof. Formula (35) is of course only a rewriting of (34). Note that for $n \geqslant 2$ and $|w| < 1$ we have

$$\frac{d}{dw} M_n = M_{n-1},$$

$$M_1(w) = \sum_{0}^{\infty} \frac{1}{j+1} w^{j+1} = -\log(1-w), \qquad |w| < 1.$$

Formulas (36)–(38) now follow by computation.

COROLLARY 12. *The function of* ζ *given by*

$$(L\mu)(\zeta) = -\zeta^n \int_K M_n(z/\zeta)\, d\mu(z)$$

extends $(L\mu)(e^{i\theta})$ *to be holomorphic on the sphere except on the union of the line segments from the origin to the points of* supp(μ).

Proof. For $z \in K$, the function $M_n(z/\zeta)$ is holomorphic on the sphere except the line segment from $\zeta = 0$ to $\zeta = z$. The conclusion now follows by integration.

THEOREM 13. *Let* α *be an open arc of the unit circle* T *which contains no point of* $K \cup \{\xi\}$. *If* $1 < p < \infty$, *then* F_σ *extends holomorphically across* α.

Proof. Let λ be a point of the arc α. According to Corollary 12 $L\mu_\sigma$ extends holomorphically to a neighborhood of λ. Further, using the notation of Proposition 11, G is actually $e^{i(n-r)\theta} M_{n-r}(\xi e^{-i\theta})$ so that G is also holomorphic in a neighborhood of λ. Now (15) and (16)(b) and standard facts from function theory (see [1]) imply that $F_\sigma^{(n)}$ has an analytic extension across T near λ and hence the same holds for F_σ.

COROLLARY 14. *If $K \cup \{\xi\} \subset \Delta_0$, then F_σ extends to be holomorphic in $\{z: |z| < R\}$ for some $R > 1$. If $p = \infty$, then $(1/\sigma) F_\sigma^{(n)}$ is a finite Blaschke product.*

Proof. The only conclusion yet to be proved is the case when $p = \infty$. Here, $L\mu_\sigma + G$ extends to be holomorphic on $\{z: |z| > t_0\}$ for some $t_0 < 1$. If $L\mu_\sigma + G + h_\sigma = 0$ on any set of positive measure in T, then $L\mu_\sigma + G + h_\sigma$ vanishes identically on T, which leads to a contradiction as in the proof of Theorem 1. Hence, $L\mu_\sigma + G + h_\sigma \neq 0$ a.e. $d\theta$. It now follows from (16)(a) and standard facts from function theory (see [1]) that $F_\sigma^{(n)}$ extends holomorphically across all of T and thus $(1/\sigma) F_\sigma^{(n)}$ is a finite Blaschke product.

Remark. Example 10 shows that in general Theorem 13 is the best to be expected since there, in the case $p = 2$, F_σ does *not* extend to holomorphic over T at any of the points of K while, of course, it does extend holomorphically acoss all other points of T.

PROPOSITION 15. *Let $K = \Delta$, $\xi = 1$, and $1 < p \leqslant \infty$, $\sigma > 0$. If F_σ is not a monomial in z and if $|F_\sigma^{(r)}(\lambda)| = \gamma(\sigma)$ for some $\lambda \in \Delta$, then λ is a root of unity.*

Proof. Define ν to be $\gamma(\sigma)/F_\sigma^{(r)}(\lambda)$ and $G(z)$ by

$$G(z) = \bar{\lambda}^r \nu F_\sigma(\lambda z), \qquad z \in \Delta.$$

Then $|G(z)| \leqslant 1$ on Δ, $\|G^{(n)}\|_p \leqslant \sigma$, and $G^{(r)}(1) = \gamma(\sigma)$. Thus, $G(z) = F_\sigma(z)$ and so

$$\lambda^{k-r} \nu = 1 \qquad \text{if} \quad F_\sigma^{(k)}(0) \neq 0.$$

Since F_σ is not a monomial, there are at least two such values of k and the proposition follows. (Note that for $p = 2$, F_σ is certainly not a monomial.)

Final remarks. (1) Clearly norms other than the sup norm over K could be imposed on f in defining the basic problem. I chose the sup norm on K for its interest and ease of formulation.

(2) It is also clear that the basic extremal problem could be formulated on a general planar domain Ω, rather than just on the unit disc Δ_0. When Ω is bounded by a finite number of disjoint smooth simple closed curves, the conclusions would be expected to follow the pattern presented here. The case of an arbitrary domain Ω is far too complex and even the solutions of simpler extremal problems are not well understood in this context.

(3) The case $p = 1$ is not handled here because if $r = n - 1$ and if $|\xi| = 1$, then G is not bounded and hence $L\mu + G$ is not in L^∞. If $0 < r < n - 1$ or if $r = n - 1$ and $|\xi| < 1$, then uniqueness of the extremal function

and the analysis of the growth of $\gamma(\sigma, 1)$ go through as above. If $p = 1$, $r = n - 1$, and $|\xi| = 1$, then G must be replaced by a jump function (with jump at ξ). However, the analysis can be altered to fit this case and again there is only one extremal function and the behavior of $\gamma(\sigma, 1)$ is like that described for $1 < p \leqslant \infty$.

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